Yiddish of the Day

"Er kukt mit di oygn,
hert mit di oyern,
un farshteyt vi di vant" =

ער קוקט מיט די
אויגן, הערט מיט די
אויערן, און פֿאַרשטייט
ווי די וואַנט.

He looks with the eyes,
hears with the ears, and
understands like the walls =

Dual Vector Spaces

Every vector space gives rise to another

→ so called "dual space"

We will see that many familiar stuff arise as this way

Def: i) Let V be a vector space over $\mathbb{F}$. Then a linear functional is a linear map

$$\varphi: V \to \mathbb{F}$$

ii) The vector space of linear functionals $= \mathcal{L}(V, \mathbb{F}) := V^*$ is called the dual space of V

ex) i) $V = M_n(\mathbb{F})$ have $tr: M_n(\mathbb{F}) \to \mathbb{F} \in M_n(\mathbb{F})^*$

$(a_{ij}) \mapsto \sum a_{ii}$

ii) $V = \mathbb{F}[t]$ have $ev_a(f(t)) := f(a)$

iii) $V = C^\infty(\mathbb{F})$ have $\int_a^b f(t)\,dt$

(or, more generally fix $g \in C^\infty(\mathbb{F})$ and define
$\int_{-\infty}^{\infty} \overline{g(t)} f(t)\,dt$)

Rmk: When we study inner-product spaces we will see "morally" where these come from

Important example

~~**Def:** Let V be vector space of $v \in V$. Define the **dual vector**, denoted $\underline{v^*} : V \to \mathbb{F}$ by~~

$$\underline{v^*}(w) = \begin{cases} 1 & w = v \\ 0 & \text{else} \end{cases}$$

look at next page

- ~~In particular~~, suppose now V is fd and let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis for V.

Then have list of vectors $(\underline{v_1^*, v_2^*, \ldots, v_n^*})$ in $\underline{V^*}$

Rmk: These vectors $v_i^* \in V^*$ are actually linearly

⌜ Given basis $v_1 \dots v_n$ for V. Define the following map

$$v_i^*(v_j) = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

$v_i^* : V \to \mathbb{F}$ is linear! ⌟

$$v_1^*(v_j) = \begin{cases} 1 & j=1 \\ 0 & \text{else} \end{cases}$$

$$v_2^*(v_j) = \begin{cases} 1 & j=2 \\ 0 & \text{else} \end{cases}$$

$\vdots$

Thm: In the setup above the vectors $(v_1^*, \ldots, v_n^*)$ are a basis for V^*

(we call this the dual basis.)

(Rmks: We know $\dim(\mathcal{L}(V,\mathbb{F})) = \dim(V)\dim\mathbb{F} = \dim V$

So we just have to check these vectors either linear-ind or are spanning vectors)

Pf) We'll prove $(v_1^*, \ldots, v_n^*)$ are LI.

Write

$$0 = c_1 v_1^* + \cdots + c_n v_n^* \text{ for } c_1 \ldots c_n \in \mathbb{F}$$

Plugin the vector v_1. We get

$$0 = c_1 v_1^*(v_1) + c_2 v_2^*(v_1) + \cdots + c_n v_n^*(v_1)$$

$$0 = c_1$$

Now repeat for all v_i to get that

$$0 = c_i v_i^*(v_i) = c_i$$

That is $c_1 = c_2 = \cdots = c_n = 0$ $\boxed{\ddot\smile}$

ex) i) $V = \mathbb{R}^3$ $\quad \mathcal{B} = (e_1, e_2, e_3)$

Compute $e_1^*\begin{pmatrix} a \\ b \\ c \end{pmatrix} := e_1^*\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = e_1^*(ae_1) + e_1^*(be_2) + e_1^*(ce_3) = ae_1^*(e_1) + \cancel{be_1^*(e_2)}^{\,0} + \cancel{ce_1^*(e_3)} = \underline{a}$

$e_2^*\begin{pmatrix} a \\ b \\ c \end{pmatrix} = b$

$e_3^*\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c$

ii) $V = \mathbb{F}_{\leq 2}[t]$ $\quad \mathcal{B} = (\overset{f_0}{\overset{\|}{1}}, \overset{f_1}{\overset{\|}{t}}, \overset{f_2}{\overset{\|}{t^2}})$

Compute

$f_0^*(a_0 + a_1 t + a_2 t^2) = f_0^*(a_0 f_0) + f_0^*(a_1 f_1) + f_0^*(a_2 f_2) = a_0$

$f_1^*(a_0 + a_1 t + a_2 t^2) = a_1$

$f_2^*(a_0 + a_1 t + a_2 t^2) = a_2$

iii) $V = M_{2\times 2}(\mathbb{R})$ $\mathcal{B} = \left(\underset{m_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{m_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{m_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{m_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right)$

Compute

$$m_1^*\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a$$

$$m_2^*\begin{pmatrix} a & b \\ c & d \end{pmatrix} = b$$

$$m_3^*\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c$$

$$m_4^*\begin{pmatrix} a & b \\ c & d \end{pmatrix} = d$$

ex) $tr : V \to \mathbb{R}$ is an element of V^*

$$tr = m_1^* + m_4^*$$

$$tr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d = m_1^*\begin{pmatrix} a & b \\ c & d \end{pmatrix} + m_4^*\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The "double dual"

Def: V vector space. Then the double dual is $(V^*)^*$ $\qquad (V^{**}) = \mathcal{L}(V^*, \mathbb{F})$

Rmk: We know that $V \cong \underline{V^*} \cong \underline{V^{**}}$

so we already know that $V \cong \underline{V^{**}}$

However! There's a "more natural" isomorphism.

(see hw about this)

Def: Let $v \in V$. Then define $\underline{ev_v} \in \underline{V^{**}}$ ("evaluation at v")
by

$$\underline{ev_v}(f) := \underline{f(v)} \quad \text{for } f \in \underline{V^*} \quad (f : V \to \mathbb{F})$$

This defines a function

$$V \xrightarrow{\Phi} \underline{V^{**}}$$
$$v \longmapsto \underline{ev_v}$$

$$(\Phi(v) = ev_v)$$

Thm: If V is fd the map $\Phi: V \to V^{**}$ is an isomorphism

Pf) We first prove Φ is linear.

$\Phi(v+w) = ev_{v+w}$. Then note, for $f \in V^*$

$ev_{v+w}(f) = f(v+w) = f(v) + f(w)$

$= ev_v(f) + ev_w(f)$

So for all $f \in V^*$, $ev_{v+w}(f) = ev_v(f) + ev_w(f)$

$\Rightarrow \Phi(v+w) = \Phi(v) + \Phi(w)$

Also note that $\Phi(cv) = ev_{cv}$ and for $f \in V^*$

$ev_{cv}(f) = f(cv) = cf(v) = c\,ev_v(f)$

Now we show Φ is an isomorphism. Since $\dim V = \dim V^{**}$

we only need to show that Φ is injective.
(by rank-nullity thm)

Lemma: If $v \neq 0$ then $\exists f \in V^*$ such that $f(v) \neq 0$.

Pf) If $v \neq 0$ then we can extend it to a basis for V. Then the dual basis vector v^* is a linear functional st $v^*(v) \neq 0$ $\square$

We want to show that if $v \neq 0$ then $\Phi(v) = ev_v \neq 0$

If $ev_v = 0$ then $\forall f \in V^*$ $\underbrace{ev_v(f)}_{= f(v)} = 0$

$ev_v = 0 \iff f(v) = 0 \quad \forall f \in V^*$. By the lemma this can't happen if $v \neq 0$. $\square$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & & \downarrow \varphi \\ & & \mathbb{F} \end{array}$$

Def: Let $T: V \to W$ be a linear map and let $\underline{\varphi} \in \underline{W^*}$

Then we get a new linear map $\underline{\varphi \circ T} : \underline{V} \to \underline{\mathbb{F}}$ called the <u>dual</u> of T defined by

$$\underline{T^*}(\varphi) = \varphi \circ T \quad \text{for } \varphi \in W^*$$

$$T^*: W^* \to V^*$$

Rmk: If $\dim V = n$, $\dim W = m$ we can identify $T \longleftrightarrow [T] \in M_{m \times n}(\mathbb{F})$

Then this new map $\underline{T^*} \longleftrightarrow [T^*] \in M_{n \times m}(\mathbb{F})$

hmmm . . . .

ex) $V = \mathbb{R}_{\leq 2}[t]$ $\quad \mathcal{B}_V = (f_0, f_1, f_2) = (1, t, t^2)$

$W = M_{2\times 2}(\mathbb{R})$ $\quad \mathcal{B}_W = (m_1, m_2, m_3, m_4)$

$T: V \to W$ by $T(a_0 + a_1 t + a_2 t^2) = \begin{pmatrix} a_0 & a_1 - a_2 \\ a_2 & 0 \end{pmatrix}$

i) Compute $[T]_{\mathcal{B}_V}^{\mathcal{B}_W}$

- $T(f_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = m_1$

- $T(f_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = m_2$ $\longrightarrow [T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

- $T(f_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -m_2 + m_3$

ii) Compute $[T^*]_{B_W^*}^{B_V^*}$

- $T^*(m_1^*) = m_1^* \circ T$

$$T^*(m_1^*)(a_0 + a_1 t + a_2 t^2)$$

$$= m_1^* \circ T(a_0 + a_1 t + a_2 t^2)$$

$$= m_1^* \begin{pmatrix} a_0 & a_1 - a_2 \\ a_2 & 0 \end{pmatrix} = a_0$$

We know that

$$T^*(m_1^*) = c_0 f_0^* + c_1 f_1^* + c_2 f_2^*$$

$$1 = T^*(m_1^*)(1) = c_0$$

$$0 = T^*(m_1^*)(t) = c_1 \quad \Rightarrow \boxed{T^*(m_1^*) = f_0^*}$$

$$0 = T^*(m_1^*)(t^2) = c_2$$

Again now compute $T^*(m_2^*) = c_0 f_0^* + c_1 f_1^* + c_2 f_2^*$

$$\rightsquigarrow \underbrace{T^*(m_2^*)}_{\in V^*}(a_0 + a_1 t + a_2 t^2) = m_2^* \begin{pmatrix} a_0 & a_1 - a_2 \\ a_2 & 0 \end{pmatrix}$$

$$= a_1 - a_2$$

$$0 = T^*(m_2^*)(1) = c_0$$

$$1 = T^*(m_2^*)(t) = c_1 \implies \boxed{T^*(m_2^*) = f_1^* - f_2^*}$$

$$-1 = T^*(m_3^*)(t^2) = c_2$$

$$T^*(m_3^*)(a_0 + a_1 t + a_2 t^2) = m_3^* \begin{pmatrix} a_0 & a_1 - a_2 \\ a_2 & 0 \end{pmatrix} = a_2$$

$\rightsquigarrow$ Find the c_0, c_1, c_2 st $T^*(m_3^*) = c_0 f_0^* + c_1 f_1^* + c_2 f_2^*$

$$0 = T^*(m_3^*)(f_0) = c_0$$

$$0 = T^*(m_3^*)(f_1) = c_1 \longrightarrow \boxed{T^*(m_3^*) = f_2^*}$$

$$1 = T^*(m_3^*)(f_2) = c_2$$

$$T^*(m_4^*)(a_0 + a_1 t + a_2 t^2) = m_4^* \begin{pmatrix} a_0 & a_1 - a_2 \\ a_2 & 0 \end{pmatrix} = 0$$

$$T^*(m_4^*) = 0 f_0^* + 0 f_1^* + 0 f_2^*$$

$$[T^*]_{\mathcal{B}_W^*}^{\mathcal{B}_V^*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

$T^* : W^* \to V^*$

$g \to g \circ T$

$$V \xrightarrow{T} W \xrightarrow{g} \mathbb{R}$$

iii) What is $[\quad]_{\mathcal{B}_W^*}^{\mathcal{B}_V^*}$

iv) Anything you notice ??

$$[T^*] = [T]^t$$

Thm: V, W finite dim with basis $\mathcal{B}_V = (v_1 \dots v_n)$

$\mathcal{B}_W = (w_1 \dots w_m)$

Then $[T^*]_{\mathcal{B}_W^*}^{\mathcal{B}_V^*} = \left([T]_{\mathcal{B}_V}^{\mathcal{B}_W} \right)^{\underline{tr}}$

Pf) Skipped

Applications

Def: Let $W \subseteq V$ subspace. Define $W^0 = \{ g : V \to \mathbb{F} \mid g(w) = 0 \ \forall w \in W \} \subseteq V^*$

and call it the annihilator of W

HW) i) If V is fd show $\dim W^0 = \dim V - \dim W$

(hint, dual basis)

ii) Use universal property of quotient to show that

$(V/W)^* \simeq W^0$

Prop: Let V, W fd and $T: V \to W$ linear. Then
$$\underbrace{\dim(\operatorname{im}(T)) = \dim(\operatorname{im}(T^*))}$$

Pf) [Skipped in Class. Proven below now.]

We claim that $\operatorname{im} T^* \subseteq (\ker T)^0$

Indeed let $\gamma \in \operatorname{im} T^*$, that is $\gamma = g \circ T$ for some $g \in W^*$.
We want to show that $\gamma(v) = 0$ $\forall v \in \ker T$.
Let $v \in \ker(T)$. Then $\gamma(v) = g(T(v)) = g(0) = 0$ ✓
Thus we have

$$\dim(\operatorname{im} T^*) \leq \dim((\ker T)^0)$$
$$\overset{HW}{=} \dim V - \dim(\ker T)$$
$$\overset{\text{rank-nullity}}{=} \dim(\operatorname{im} T).$$

Now we can do the same thing to show

that $\dim(\operatorname{im}(T^*)^*) \le \dim(\operatorname{im} T^*)$

However T^{**} is really just T under the isomorphism

$\Phi : V \xrightarrow{\sim} V^{**}$. So $\dim(\operatorname{im} T) = \dim(\operatorname{im} T^{**}) \le \dim(\operatorname{im} T^*)$

Thus $\dim(\operatorname{im} T) = \dim(\operatorname{im} T^*)$ $\square$

<u>Cor</u>: For a matrix A <u>row rank A</u> $=$ <u>col rank A</u>

Pf) Special case of $T : \mathbb{F}^n \to \mathbb{F}^m$. Then we saw

T^* is just the transpose of T. So

$$\text{row rank} = \dim(\operatorname{col}(T)^{tr})$$
$$= \dim(\operatorname{im} T^*)$$

$= \dim(\operatorname{im} T) = \dim(\operatorname{col}(T)) = \operatorname{col} \operatorname{rank}$ $\square$